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Bounds for triple zeta-functions

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ABSTRACT

In the present paper we consider the problem of the order of magnitude for the triple zeta-functions of Euler–Zagier type in the region $0 < \Re s_j < 1$ $(j = 1, 2, 3)$. We apply the Euler–Maclaurin summation formula and van der Corput's method of multiple exponential sums to the triple zeta-functions.

1 INTRODUCTION

Let $s_j = \sigma_j + it_j$ $(j = 1, 2, \ldots, r)$ be complex variables. The r-ple zeta-function of Euler–Zagier type is defined by

$$\zeta_r(s_1, \ldots, s_r) = \sum_{1 \leqslant n_1 < \cdots < n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}},$$

which is absolutely convergent for $\sigma_r > 1, \sigma_r + \sigma_{r-1} > 2, \ldots, \sigma_r + \cdots + \sigma_1 > r$. This function has many applications to mathematical physics. In particular, algebraic relations among the values of $\zeta_r(s_1, \ldots, s_r)$ at positive integers have been studied extensively (cf. [10]). As a function of complex variables s_j, the analytic continuation of $\zeta_r(s_1, \ldots, s_r)$ has already been established. In fact, the case $r = 2$ was studied by Atkinson [3] in his research on the mean value formula of the Riemann zeta-function. For general r, the analytic continuation of $\zeta_r(s_1, \ldots, s_r)$ was proved

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by Akiyama, Egami and Tanigawa [1] and Zhao [12] independently, and later by Matsumoto [9]. The values at negative integers were considered in [1] and [2].

On the other hand, the order of magnitude of relevant zeta-functions on some vertical line plays an important role in the theory of numbers. In this respect, we are naturally led to the study of such a problem for $\zeta_r(s_1, \dots, s_r)$. Ishikawa and Matsumoto [5] first obtained some results in this direction. Before stating their results, we write down the definitions of these two zeta-functions (in the domain of absolute convergence) since we do not treat the general one in this paper:

$$\zeta_2(s_1, s_2) = \sum_{1 \leqslant n_1 < n_2} \frac{1}{n_1^{s_1} n_2^{s_2}} \tag{1.1}$$

and

$$\zeta_3(s_1, s_2, s_3) = \sum_{1 \leqslant n_1 < n_2 < n_3} \frac{1}{n_1^{s_1} n_2^{s_2} n_3^{s_3}}. \tag{1.2}$$

Though Ishikawa and Matsumoto treated the general r-ple zeta function, they also considered double and triple zeta-functions separately. Their assertions on these functions are

$$\zeta_2(it, i\alpha t) \ll (1 + |t|)^{3/2+\varepsilon} \tag{1.3}$$

and

$$\zeta_3(-it, it, it) \ll (1 + |t|)^{5/2+\varepsilon}, \tag{1.4}$$

where α ($\neq \pm 1$) is a fixed constant and any $\varepsilon > 0$, by taking the absolute value of the Mellin–Barnes integral representation of the multiple zeta function.

Recently, in [7], we improved their results in the case of double zeta function by using the Euler–Maclaurin summation formula and van der Corput's method of double exponential sums due to Krätzel [8]. Our result reads as follows.

Theorem A. *Let $|t_1|$ and $|t_2| \geqslant 2$ be real numbers such that $|t_1| \asymp |t_2|$ and $|t_1 + t_2| \gg 1$.*

In the case $\sigma_1 = \sigma_2 = 0$, we have

$$\zeta_2(it_1, it_2) \ll |t_1| \log^2 |t_1|. \tag{1.5}$$

Suppose that $0 \leqslant \sigma_j < 1$ and $\sigma_1 + \sigma_2 > 0$. Then we have

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) \tag{1.6}$$
$$\ll \begin{cases} |t_1|^{1-\frac{2}{3}(\sigma_1+\sigma_2)} \log^2 |t_1| & (0 \leqslant \sigma_1 \leqslant \frac{1}{2}, 0 \leqslant \sigma_2 \leqslant \frac{1}{2}) \\ |t_1|^{\frac{5}{6}-\frac{1}{3}(\sigma_1+2\sigma_2)} \log^3 |t_1| & (\frac{1}{2} < \sigma_1 < 1, 0 \leqslant \sigma_2 \leqslant \frac{1}{2}) \\ |t_1|^{\frac{5}{6}-\frac{1}{3}(2\sigma_1+\sigma_2)} \log^3 |t_1| & (0 \leqslant \sigma_1 \leqslant \frac{1}{2}, \frac{1}{2} < \sigma_2 < 1) \\ |t_1|^{\frac{2}{3}-\frac{1}{3}(\sigma_1+\sigma_2)} \log^4 |t_1| & (\frac{1}{2} < \sigma_1 < 1, \frac{1}{2} < \sigma_2 < 1). \end{cases}$$

On the "critical line" $\sigma_1 = \sigma_2 = 1/2$ our result is as sharp as the Hardy–Littlewood bound $\mu(1/2) \leqslant 1/6$ for the Riemann zeta-function, where $\mu(\sigma)$ $(0 \leqslant \sigma \leqslant 1)$ denote the infimum of a number c such that

$$\zeta(\sigma + it) \ll |t|^c.$$

Furthermore, under the same conditions on $|t_j|$ $(j = 1, 2)$, we conjectured that

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) \ll |t_1|^{\mu(\sigma_1)+\mu(\sigma_2)} \log^A |t_1|$$

with some constant A.

In this paper, we shall consider the order of magnitude for the triple zeta-functions of Euler–Zagier type. Our main purpose is to prove the following theorem.

Theorem. *Let $|t_j| \geqslant 2$ $(j = 1, 2, 3)$ be real numbers such that*

$$|t_1| \asymp |t_2| \asymp |t_3|$$

and

$$|t_2 + t_3| \gg 1, |t_1 + t_2 + t_3| \gg 1.$$

In the case $\sigma_1 = \sigma_2 = \sigma_3 = 0$, we have

$$\zeta_3(it_1, it_2, it_3) \ll |t_1|^2 \log^2 |t_1|. \tag{1.7}$$

Suppose that $0 \leqslant \sigma_j < 1$ and $\sigma_1 + \sigma_2 + \sigma_3 > 0$. Then we have

$$\zeta_3(s_1, s_2, s_3) \ll |t_1|^{2-\frac{2}{3}(\sigma_1+\sigma_2+\sigma_3)} \log^3 |t_1|. \tag{1.8}$$

It is easily seen that

$$\begin{aligned}\zeta_3(it, it, it) &= \frac{1}{6}\big(\zeta(it)^3 - \zeta(3it) - 3\zeta_2(2it, it) - 3\zeta(it, 2it)\big)\\ &\ll |t|^{3/2+\varepsilon}.\end{aligned}$$

Hence we can expect that

$$\zeta_3(it_1, it_2, it_3) \ll |t_1|^{\frac{3}{2}} \log^B |t_1| \tag{1.9}$$

under the same conditions of Theorem, where B is a constant. Though the formula (1.7) improves Ishikawa and Matsumoto's bound (1.4), it is worse by 1/2 than the conjectured bound (1.9). The reason is that we use Krätzel's theorem on multiple exponential sum which is derived by induction starting from the double exponential sum. In order to improve (1.7), it is necessary to develop the theory of *truly* triple exponential sum.

We can also conjecture that

$$\zeta_3(\sigma_1+it_1,\sigma_2+it_2,\sigma_3+it_3) \ll |t_1|^{\mu(\sigma_1)+\mu(\sigma_2)+\mu(\sigma_3)} \log^C |t_1| \tag{1.10}$$

under the same conditions of Theorem, where C is a constant.

2 PRELIMINARIES

To prove our theorem, we quote here the following lemmas (see Corollary 2 and Lemma 3 of [7]).

Lemma 1. *Let $s=\sigma+it$ and $|t|>1$. For $N>\frac{1}{4}|t|$ and $\sigma>-3$, we have*

$$\zeta(s) = \sum_{n\leqslant N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{N^{-s}}{2} + \frac{s}{12}N^{-s-1} + O\left(|t|^3 N^{-\sigma-3}\right),$$

where the implied constant does not depend on t.

Lemma 2. *Let $t>2$, $N\leqslant N_1\leqslant 2N$ and $N\ll t$, then we have*

$$\sum_{N<n\leqslant N_1}\frac{1}{n^{1/2+it}} \ll t^{\frac{1}{6}}, \tag{2.1}$$

$$\sum_{N<n\leqslant N_1}\frac{1}{n^{it}} \ll t^{\frac{1}{2}}, \tag{2.2}$$

$$\sum_{N<n\leqslant N_1}\frac{1}{n^{s}} \ll \begin{cases} t^{\frac{1}{2}-\frac{2}{3}\sigma} & (0<\sigma<\frac{1}{2}) \\ t^{\frac{1}{3}-\frac{1}{3}\sigma}\log t & (\frac{1}{2}<\sigma\leqslant 1) \end{cases} \tag{2.3}$$

and

$$\zeta(\sigma+it) \ll \begin{cases} t^{\frac{1}{2}-\frac{2}{3}\sigma}\log t & (0\leqslant\sigma\leqslant\frac{1}{2}) \\ t^{\frac{1}{3}-\frac{1}{3}\sigma}\log^2 t & (\frac{1}{2}<\sigma\leqslant 1). \end{cases} \tag{2.4}$$

Let $s_j=\sigma_j+it_j$ $(j=1,2)$ be complex variables with $|t_1|\asymp|t_2|$. We take a parameter τ such that $\max\{|t_1|,|t_2|,|t_1+t_2|\}+2\leqslant\tau\ll|t_1|$.

Assuming that $\sigma_j>1$ $(j=1,2)$, we divide the double series (1.1) as

$$\begin{aligned} \zeta_2(s_1,s_2) &= \sum_{m<n\leqslant\tau}\frac{1}{m^{s_1}n^{s_2}} + \sum_{\substack{m<n\\ n>\tau}}\frac{1}{m^{s_1}n^{s_2}} \\ &=: S_1^{(2)}(s_1,s_2;\tau) + S_2^{(2)}(s_1,s_2;\tau), \end{aligned} \tag{2.5}$$

say. Furthermore, we define, for $M<N<2M$,

$$T_2(s_1,s_2;M,N) = \sum_{M<n_1<n_2\leqslant N}\frac{1}{n_1^{s_1}n_2^{s_2}}. \tag{2.6}$$

After analytic continuation of $S_j^{(2)}(s_1, s_2; \tau)$ we can estimate the order of magnitude of these sums in the range $0 \leqslant \sigma_j < 1$ $(j = 1, 2)$. Recently, we obtained the upper bounds of the above sums $S_j^{(2)}(s_1, s_2; \tau)$ and $T_2(s_1, s_2; M, N)$ in the range $0 \leqslant \sigma_j < 1$ $(j = 1, 2)$ [7].

Lemma 3. *Under the conditions above, we have*

$$S_1^{(2)}(it_1, it_2; \tau) \ll \tau \log^2 \tau. \tag{2.7}$$

Furthermore, suppose that $0 \leqslant \sigma_j < 1$ $(j = 1, 2)$, $\sigma_1 + \sigma_2 > 0$. *Then we have*

$$T_2(s_1, s_2; M, N) \ll \begin{cases} M^{-\sigma_1-\sigma_2}\tau \log \tau & (\tau^{2/3} \ll M \ll \tau) \\ M^{1-\sigma_1-\sigma_2}\tau^{1/3} \log^{1/2} \tau & (\tau^{1/3} \ll M \ll \tau^{2/3}) \\ M^{2-\sigma_1-\sigma_2} \log \tau & (M \ll \tau^{1/3}) \end{cases} \tag{2.8}$$

and

$$\sum_{j \geqslant 1} T_2\big(s_1, s_2; 2^{-j}\tau, 2^{-j+1}\tau\big) \ll \big(\tau^{1-\frac{2}{3}(\sigma_1+\sigma_2)} + \tau^{\frac{2}{3}-\frac{1}{3}(\sigma_1+\sigma_2)}\big) \log^2 \tau. \tag{2.9}$$

Hence we have

$$S_1^{(2)}(s_1, s_2; \tau) \ll \begin{cases} \tau^{1-\frac{2}{3}(\sigma_1+\sigma_2)} \log^2 \tau & (0 \leqslant \sigma_1 \leqslant \frac{1}{2}, 0 \leqslant \sigma_2 \leqslant \frac{1}{2}) \\ \tau^{\frac{5}{6}-\frac{1}{3}(\sigma_1+2\sigma_2)} \log^3 \tau & (\frac{1}{2} < \sigma_1 < 1, 0 \leqslant \sigma_2 \leqslant \frac{1}{2}) \\ \tau^{\frac{5}{6}-\frac{1}{3}(2\sigma_1+\sigma_2)} \log^3 \tau & (0 \leqslant \sigma_1 \leqslant \frac{1}{2}, \frac{1}{2} < \sigma_2 < 1) \\ \tau^{\frac{2}{3}-\frac{1}{3}(\sigma_1+\sigma_2)} \log^4 \tau & (\frac{1}{2} < \sigma_1 < 1, \frac{1}{2} < \sigma_2 < 1) \end{cases} \tag{2.10}$$

and

$$S_2^{(2)}(s_1, s_2; \tau) \ll \tau^{1-(\sigma_1+\sigma_2)}. \tag{2.11}$$

We also recall the partial summation formula for the double sum $\sum_{M < m \leqslant n \leqslant N} f(m, n) g(m, n)$, where M and N are positive integers such that $M < N$, $f(x, y)$ is a C^2-function on $[M, N]^2$ and $g(m, n)$ is an arithmetical function on the same domain [7].

Lemma 4. *Under the notation as above, suppose that*

$$G(x, y) = \sum_{x < m \leqslant n \leqslant y} g(m, n)$$

and

$$|G(x, y)| \leqslant G, \qquad |f_x(x, y)| \leqslant \kappa_1,$$
$$|f_y(x, y)| \leqslant \kappa_2, \qquad |f_{xy}(x, y)| \leqslant \kappa_3$$

for any $(x, y) \in [M, N]^2$.

Then we have the equation

$$(2.12)\qquad \sum_{M<m\leqslant n\leqslant N} f(m,n)g(m,n)$$
$$= f(M,N)G(M,N) + \int_M^N \big(f_x(x,N)g(x,N) - f_y(x,x)G(M,x)\big)\,dx$$
$$+ \int_M^N \int_x^N f_{xy}(x,y)\big(G(M,y) - G(x,y)\big)\,dy\,dx.$$

Furthermore we have

$$(2.13)\qquad \Bigg|\sum_{M<m\leqslant n\leqslant N} f(m,n)g(m,n)\Bigg|$$
$$\leqslant G\big(|f(M,N)| + (\kappa_1+\kappa_2)(N-M) + \kappa_3(N-M)^2\big).$$

For our purpose we need the partial summation formula for the triple sum

$$\sum_{M<l\leqslant m\leqslant n\leqslant N} \alpha(l,m,n)\beta(l,m,n),$$

where $\alpha(x,y,z)$ is a C^3-function on $[M,N]^3$ and $\beta(l,m,n)$ is an arithmetica function on the same domain. Define

$$B(x,y,z) = \sum_{x<l\leqslant m\leqslant y} \beta(l,m,z)$$

and

$$B_1(M,x) = \sum_{M<n\leqslant x} B(M,n,n),$$
$$B_2(x,y) = \sum_{x<n\leqslant y} B(M,x,n),$$
$$B_3(y,z) = \sum_{y<n\leqslant z} B(x,y,n)$$

for any $(x,y,z)\in[M,N]^3$. We can prove the following lemma.

Lemma 5. *Under the above notations, suppose that*

$$\begin{array}{lll}
|B_1(M,x)|\leqslant B, & |B_2(x,y)|\leqslant B, & |B_3(y,z)|\leqslant B,\\
|\alpha(x,y,z)|\leqslant\lambda_0, & |\alpha_x(x,y,z)|\leqslant\lambda_1, & |\alpha_y(x,y,z)|\leqslant\lambda_2,\\
|\alpha_z(x,y,z)|\leqslant\lambda_3, & |\alpha_{xy}(x,y,z)|\leqslant\lambda_4, & |\alpha_{yz}(x,y,z)|\leqslant\lambda_5,\\
|\alpha_{zx}(x,y,z)|\leqslant\lambda_6, & |\alpha_{xyz}(x,y,z)|\leqslant\lambda_7 &
\end{array}$$

for any $(x, y, z) \in [M, N]^3$.

Then we have

$$(2.14)\qquad \left|\sum_{M<l\leqslant m\leqslant n\leqslant N} \alpha(l,m,n)\beta(l,m,n)\right| \\ \ll B\big(\lambda_0 + (\lambda_1+\lambda_2+\lambda_3)(N-M) \\ + (\lambda_4+\lambda_5+\lambda_6)(N-M)^2 + \lambda_7(N-M)^3\big).$$

Proof. The triple sum of the above can be written as

$$(2.15)\qquad \sum_{M<l\leqslant m\leqslant n\leqslant N} \alpha(l,m,n)\beta(l,m,n) = \sum_{M<n\leqslant N}\sum_{M<l\leqslant m\leqslant n} \alpha(l,m,n)\beta(l,m,n) \\ = \sum_{M<n\leqslant N} V(M,n),$$

say. We shall employ the similar method used for equation (2.12). The double sum for $V(M,n)$ can be written as, for a fixed positive integer n,

$$V(M,n) = \alpha(M,n,n)\beta(M,n,n) \\ + \int_M^n \big(\alpha_x(x,n,n)B(x,n,n) - \alpha_y(x,x,n)B(M,x,n)\big)\,dx \\ + \int_M^n\int_x^n \alpha_{xy}(x,y,n)\big(B(M,y,n) - B(x,y,n)\big)\,dy\,dx \\ = V_1(M,n) + V_2(M,n) + V_3(M,n),$$

say. Firstly, we have

$$\sum_{M<n\leqslant N} V_1(M,n) = \alpha(M,N,N)B_1(M,N) \\ - \int_M^N \big(\alpha_y(M,x,x) + \alpha_z(M,x,x)\big)B_1(M,x)\,dx.$$

We put

$$\sum_{M<n\leqslant N} V_2(M,n) = \int_M^N \sum_{x<n\leqslant N} \alpha_x(x,n,n)B(x,n,n)\,dx \\ - \int_M^N \sum_{x<n\leqslant N} \alpha_y(x,x,n)B(M,x,n)\,dx \\ = U_1(M,N) - U_2(M,N),$$

say. We use partial summation formula to get

$$U_1(M,N)=\int_M^N \alpha_x(x,N,N)B_1(x,N)\,dx - \int_M^N\int_x^N \big(\alpha_{xy}(x,y,y)+\alpha_{xz}(x,y,y)\big)B_1(x,y)\,dy\,dx,$$

and

$$U_2(M,N)=\int_M^N \alpha_y(x,x,N)B_2(x,N)\,dx-\int_M^N\int_x^N \alpha_{yz}(x,x,y)B_2(x,z)\,dz\,dx$$

Hence we have

$$\sum_{M<n\leqslant N} V_2(M,n)=\int_M^N \big(\alpha_x(x,N,N)B_1(x,N)-\alpha_y(x,x,N)B_2(x,N)\big)\,dx - \int_M^N\left\{\int_x^N \big(\alpha_{xy}(x,y,y)+\alpha_{zx}(x,y,y)\big)B_1(x,y)\,dy - \int_x^N \alpha_{yz}(x,x,z)B_2(x,z)\,dz\right\}dx. \tag{2.16}$$

We put

$$\begin{aligned}\sum_{M<n\leqslant N} V_3(M,n)&=\int_M^N\int_x^N \sum_{y<n\leqslant N}\alpha_{xy}(x,y,n)B(M,y,n)\,dx\\&\quad-\int_M^N\int_x^N \sum_{y<n\leqslant N}\alpha_{xy}(x,y,n)B(x,y,n)\,dy\,dx\\&=U_3(M,N)-U_4(M,N),\end{aligned}$$

say. We obtain

$$U_3(M,N)=\int_M^N\int_x^N \alpha_{xy}(x,y,N)B_2(y,N)\,dy\,dx - \int_M^N\int_x^N\int_y^N \alpha_{xyz}(x,y,z)B_2(y,z)\,dz\,dy\,dx$$

and

$$U_4(M,N)=\int_M^N\int_x^N \alpha_{xy}(x,y,N)B_3(y,N)\,dy\,dx - \int_M^N\int_x^N\int_y^N \alpha_{xyz}(x,y,z)B_3(y,z)\,dz\,dy\,dx.$$

Hence we have

$$\sum_{M<n\leqslant N} V_3(M,n)=\int_M^N\int_x^N \alpha_{xy}(x,y,N)\big(B_2(y,N)-B_3(y,N)\big)\,dy\,dx - \int_M^N\int_x^N\int_y^N \alpha_{xyz}(x,y,z)\big(B_2(y,z)-B_3(y,z)\big)\,dz\,dy\,dx. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15), we take absolute value in the right-hand side, thus we get the assertion (2.14). □

3 KRÄTZEL'S THEORY OF MULTIPLE EXPONENTIAL SUMS

Next we shall recall the result for the multiple exponential sums, which is given by Krätzel [8]. To state his lemma, we assume the following conditions (see [8]):

(I) Let D be a bounded p-dimensional domain ($p\geqslant 2$) with a volume $|D|$, where the number of lattice points are of order $|D|$. Suppose that D is a subset of the hyper-rectangle

$$E=\{\mathbf{x}\mid \mathbf{x}=(x_1,x_2,\dots,x_p),\, a_j\leqslant x_j\leqslant b_j\ (j=1,2,\dots,p)\}$$

with $|E|=\prod_{j=1}^p c_j,\ c_j=b_j-a_j\ (j=1,2,\dots,p)$.

(II) Any straight line parallel to any of the coordinate axes intersects D in a bounded number of line segment. For the sake of simplicity we only consider such domains D where these straight lines intersect the boundary of D in at most two points or in one line segment. We can assume this without loss of generality, because each such general domain can be divided into a finite number of these special domains.

(III) Let $h(\mathbf{x})$ be real function in E with continuous partial derivatives of as many order as may be required. Suppose that the functions $h_{x_j}(\mathbf{x})$ are monotonic with respect to $x_j\ (j=1,2,\dots,p)$. Intersections of D with domains of the type $h_{x_j}(\mathbf{x})\leqslant c$ or $h_{x_j}(\mathbf{x})\geqslant c\ (j=1,2,\dots,p)$ are to satisfy condition (II) as well, where c is a constant.

(IV) The boundary of D can be divided into a bounded number of parts. In each part the boundary is given by

$$x_j=r_j(x_{j+1},\dots,x_p)\quad \text{for } j=2,3,\dots,p-1$$

or

$$x_p = c \quad \text{or} \quad x_1 = \rho(x_2, x_3, \ldots, x_p)$$

with continuous partial derivatives of as many orders as may be required.

(V) Let D_j $(j = 1, 2, \ldots, p)$ be the image of D under the mapping

$$y_\nu = \begin{cases} h_{x_\nu}(\mathbf{x}) & \text{for } \nu = 1, 2, \ldots, j \\ x_\nu & \text{for } \nu = j+1, j+2, \ldots, p. \end{cases}$$

Suppose that the number of lattice points of D_j is of order of the volume $|D_j|$. Furthermore, we put $D_0 = D$. Let E_μ $(\mu = 1, 2, \ldots, p)$ denote the hyperplane $y_\mu = c$. We consider the intersections $D_j \cap E_\mu$ $(j = 0, 1, \ldots, p)$ for each constant c. Suppose that the greatest intersection is contained in a domain $R_{j,\mu}$ with

$$1 \ll |R_{j,\mu}| \ll |D_j|.$$

(VI) Let the functions $\varphi_\nu(\mathbf{y})$ and $F_j(\mathbf{y})$ with $\mathbf{y} = (y_1, y_2, \ldots, y_p)$ be defined by

$$h_{x_\nu}(\varphi_1, \ldots, \varphi_j, y_{j+1}, \ldots, y_p) = y_\nu \quad \text{for } \nu = 1, 2, \ldots, p,$$

and

$$F_j(\mathbf{y}) = h(\varphi_1, \ldots, \varphi_j, y_{j+1}, \ldots, y_p) - \sum_{\nu=1}^{j} y_\nu \varphi_\nu(\mathbf{y}) \quad \text{for } j = 1, 2, \ldots, p.$$

Furthermore, we put $F_0(\mathbf{y}) = h(\mathbf{y})$.

(VII) Suppose that for each point $\mathbf{v} \in D_j$ with $\mathbf{v} = (v_1, v_2, \ldots, v_p)$, the functions

$$\left(\frac{\partial}{\partial y_{j+1}} F_j(\mathbf{y}_j) - \frac{\partial}{\partial y_{j+1}} F_j(\mathbf{v})\right)^6 - 8 \frac{\partial^2}{\partial y_{j+1}^2} F_j(\mathbf{v}) \left(\frac{\partial^2}{\partial y_{j+1}^2} F_j(\mathbf{y}_j)\right)^2 \\ \times \left\{F_j(\mathbf{y}_j) - F_j(\mathbf{v}) - (\mathbf{y}_{j+1} - \mathbf{v}) \frac{\partial}{\partial y_{j+1}} F_j(\mathbf{v})\right\}^6$$

have only a bounded number of points of zeros, where

$$\mathbf{y}_j = (v_1, \ldots, v_j, y_{j+1}, v_{j+2}, \ldots, v_p) \quad \text{for } j = 0, 1, \ldots, p-1.$$

Hence, we formulate the following lemma.

Lemma 6 (Theorem 2.29 in [8]). *Let the Hessian $H_j(h)$ $(j = 1, 2, \ldots, p)$ be defined by the functional determinant*

$$H_j(h) = \frac{\partial(h_{x_1}, h_{x_2}, \ldots, h_{x_j})}{\partial(x_1, x_2, \ldots, x_j)} \quad \textit{and} \quad H_0(h) = 1.$$

Suppose that

$$\Lambda_0 = 1, \qquad |H_j(h)| \asymp \Lambda_j,$$
$$\left|\frac{\partial^3}{\partial y_j^3} F_{j-1}(\mathbf{y})\right| \asymp L_j, \quad L_j \ll \lambda_j \ll \left(\frac{\Lambda_j}{\Lambda_{j-1}}\right)^2$$

for $j = 1, 2, \ldots, p$. *Here* L_j *may depend on* $y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_p$. *Assume the functions*

$$u_j(\mathbf{y}) = \frac{\partial}{\partial y_j} \left|H_{j-1}\big(h(\varphi_1, \ldots, \varphi_{j-1}, y_j, \ldots, y_p)\big)\right|^{-\frac{1}{2}}$$

to be monotonic with respect to the y_j, *and let*

$$|u_j(\mathbf{y})| \asymp G_j.$$

Suppose that at the boundary of D *partly either* $h_{x_1}(\rho(x_2, \ldots, x_p), x_2, \ldots, x_p)$ *is an integer, a constant not depending on the parameters of the problem or*

$$\left|\frac{\partial}{\partial x_2} h_{x_1}\big(\rho(x_2, \ldots, x_p), x_2, \ldots, x_p\big)\right| \gg \sqrt{\Lambda_2}.$$

Then we have

$$\sum_{(n_1, \ldots, n_p) \in D} e\big(h(n_1, \ldots, n_p)\big) \ll |D|\sqrt{\Lambda_p} + \Delta \tag{3.1}$$

with

$$\Delta = \frac{|R_{2,1}| + |R_{1,2}| \log(c_2 + 1)}{\sqrt{\Lambda_2}} + \sum_{j=3}^{p} \frac{|R_{j-1,j}|}{\sqrt{\Lambda_j}} + \sum_{j=1}^{p} \frac{|R_{j-1,j}|}{\sqrt{\Lambda_{j-1}}} \log\left(\frac{c_j \Lambda_j}{\Lambda_{j-1}} + 2\right).$$

4 PROOF OF THEOREM

4.1. Decomposition of the triple series (1.2)

Let $s_j = \sigma_j + it_j$ $(j = 1, 2, 3)$ be complex variables with $|t_1| \asymp |t_2| \asymp |t_3|$. In the sequel of this paper, we take a parameter τ such that

$$2\max\{|t_1|, |t_2|, |t_3|, |t_1 + t_2|, |t_2 + t_3|, |t_1 + t_2 + t_3|\} = \tau \ll |t_1|.$$

Firstly assume that $\sigma_3 > 1, \sigma_2 + \sigma_3 > 2$ and $\sigma_1 + \sigma_2 + \sigma_3 > 3$, in which region the triple series (1.2) is absolutely convergent. We divide the series (1.2) as

$$\begin{aligned}(4.1)\qquad \zeta_3(s_1,s_2,s_3) &= \sum_{n_1<n_2<n_3\leqslant\tau}\frac{1}{n_1^{s_1}n_2^{s_2}n_3^{s_3}} + \sum_{\substack{n_1<n_2<n_3\\ n_3>\tau}}\frac{1}{n_1^{s_1}n_2^{s_2}n_3^{s_3}}\\ &= S_1^{(3)}(s_1,s_2,s_3;\tau) + S_2^{(3)}(s_1,s_2,s_3;\tau),\end{aligned}$$

say. After analytic continuation of the infinite sum $S_2^{(3)}(s_1,s_2,s_3;\tau)$, we shall consider the order of magnitude of these sums in the range $0\leqslant\sigma_j<1$ $(j=1,2,3)$.

4.2. Evaluation of $S_1^{(3)}(s_1,s_2,s_3;\tau)$

(I) Firstly we consider the case $\sigma_1=\sigma_2=\sigma_3=0$. To evaluate $S_1^{(3)}(it_1,it_2,it_3;\tau)$, we apply the theory of Krätzel [8] on multiple exponential sums. Let $M\leqslant\frac{\tau}{2}$ and

$$D(M)=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid 0<x_1<x_2<x_3,\ M<x_3\leqslant 2M\right\}.$$

We consider an upper bound for the sum

$$(4.2)\qquad V_M(it_1,it_2,it_3)=\sum_{(n_1,n_2,n_3)\in D(M)\cap\mathbb{Z}^3} n_1^{it_1}n_2^{it_2}n_3^{it_3}.$$

To estimate this sum, we divide the domain $D(M)$ as

$$D(M)=D^{(1)}(M)\cup D^{(2)}(M)\cup D^{(3)}(M),$$

where

$$\begin{aligned}D^{(1)}(M)&=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1<x_2\leqslant M,\ M<x_3\leqslant 2M\right\},\\ D^{(2)}(M)&=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1\leqslant M,\ M<x_2<x_3\leqslant 2M\right\}\end{aligned}$$

and

$$D^{(3)}(M)=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid M<x_1<x_2<x_3\leqslant 2M\right\}.$$

The exponential sum on the domain $D^{(3)}(M)$ is evaluated by Lemma 6. In our present case, we have the following evaluations:

$$\begin{aligned}&h(x_1,x_2,x_3)=\frac{1}{2\pi}(t_1\log x_1+t_2\log x_2+t_3\log x_3),\\ &H_j(h)\asymp\Lambda_j=\frac{\tau^j}{M^{2j}}\quad (j=1,2,3),\\ &R_{1,2}\ll\tau,\qquad R_{2,1}\ll\tau,\qquad R_{2,3}\ll\left(\frac{\tau}{M}\right)^2,\\ &R_{0,1}\ll M^2\quad\text{and}\quad \left|D^{(3)}(M)\right|\asymp M^3.\end{aligned}$$

Hence

$$\Delta \ll \frac{\tau + \tau \log \tau}{\tau/M^2} + \frac{(\tau/M)^2}{\tau^{3/2}/M^3} + \left(M^2 + \frac{\tau}{\sqrt{\tau/M^2}} + \frac{(\tau/M)^2}{\sqrt{\tau^2/M^4}}\right) \log \tau$$
$$\ll M^2 \log \tau + \tau^{\frac{1}{2}} M + \tau \log \tau.$$

Therefore we have, from (3.1),

$$\text{(4.3)} \qquad \sum_{(n_1,n_2,n_3) \in D^{(3)}(M) \cap \mathbb{Z}^3} n_1^{it_1} n_2^{it_2} n_3^{it_3} \ll \tau^{\frac{3}{2}} + M^2 \log \tau + \tau^{\frac{1}{2}} M + \tau \log \tau$$
$$\ll \tau^2 \log \tau.$$

As for the sum on the domains $D^{(1)}(M)$ and $D^{(2)}(M)$, we use Lemma 3 and estimate of the exponential sum (cf. p. 107 in [11]) to obtain

$$\text{(4.4)} \qquad \sum_{D^{(1)}(M) \cap \mathbb{Z}^3} n_1^{it_1} n_2^{it_2} n_3^{it_3} = \sum_{M < n_3 \leqslant 2M} n_3^{it_3} \cdot \sum_{n_1 < n_2 \leqslant M} n_1^{it_1} n_2^{it_2} \ll \tau^{\frac{3}{2}} \log^2 \tau$$

and

$$\text{(4.5)} \qquad \sum_{D^{(2)}(M) \cap \mathbb{Z}^3} n_1^{it_1} n_2^{it_2} n_3^{it_3} = \sum_{n_1 < M} n_1^{it_1} \cdot \sum_{M < n_2 < n_3 \leqslant 2M} n_2^{it_2} n_3^{it_3} \ll \tau^{\frac{3}{2}} \log^3 \tau.$$

Therefore, substituting (4.3)–(4.5) into (4.2), we have

$$V_M(it_1, it_2, it_3) \ll \tau^2 \log \tau.$$

From the above estimation, we have

$$\text{(4.6)} \qquad S_1^{(3)}(it_1, it_2, it_3; \tau) = \sum_{j=1}^{[\frac{\log \tau}{\log 2}]} \overline{V_{\tau/2^j}(it_1, it_2, it_3)}$$
$$\ll \tau^2 \log^2 \tau.$$

(II) Secondly we consider the case $\sigma_1 + \sigma_2 + \sigma_3 > 0$. For this purpose we introduce two auxiliary functions. Let $0 \leqslant \sigma_j < 1$ $(j = 1, 2, 3)$ and $M < N < 2M \leqslant \tau$ and define $T_3(s_1, s_2, s_3; M, N)$ and $U_j(s_1, s_2, s_3; M)$ $(j = 1, 2)$ as

$$\text{(4.7)} \qquad T_3(s_1, s_2, s_3; M, N) = \sum_{M < n_1 < n_2 < n_3 \leqslant N} \frac{1}{n_1^{s_1} n_2^{s_2} n_3^{s_3}},$$

$$\text{(4.8)} \qquad U_1(s_1, s_2, s_3; M) = \sum_{n_1 < M < n_2 < n_3 \leqslant 2M} \frac{1}{n_1^{s_1} n_2^{s_2} n_3^{s_3}}$$
$$= \left(\sum_{n_1 < M} \frac{1}{n_1^{s_1}}\right) T_2(s_2, s_3; M, 2M)$$

and

$$(4.9)\qquad U_2(s_1,s_2,s_3;M)=\sum_{n_1<n_2\leqslant M<n_3\leqslant 2M}\frac{1}{n_1^{s_1}n_2^{s_2}n_3^{s_3}}$$
$$=S_1^{(2)}(s_1,s_2;M)\Bigl(\sum_{M<n_3\leqslant 2M}\frac{1}{n_3^{s_3}}\Bigr),$$

respectively, where the function T_2 and $S_1^{(2)}$ are defined by (2.5) and (2.6). Then we can write $S_1^{(3)}(s_1,s_2,s_3;\tau)$ as

$$(4.10)\qquad S_1^{(3)}(s_1,s_2,s_3;\tau)=\sum_{j\geqslant 1}\bigl\{T_3\bigl(s_1,s_2,s_3;2^{-j}\tau,2^{-j+1}\tau\bigr)$$
$$+U_1\bigl(s_1,s_2,s_3;2^{-j}\tau\bigr)+U_2\bigl(s_1,s_2,s_3;2^{-j}\tau\bigr)\bigr\}.$$

Let $j_0=[\log\tau/3\log 2]$ and $L=2^{-j_0}\tau\asymp\tau^{2/3}$. To reduce the estimate of $T_3(\sigma_1+it_1,\sigma_2+it_2,\sigma_3+it_3;M,N)$ for $L\ll M\ll\tau$ into that of $T_3(it_1,it_2,it_3;M,N)$, we apply Lemma 5 with

$$\alpha(x,y,z)=\frac{1}{x^{\sigma_1}y^{\sigma_2}z^{\sigma_3}}$$

and

$$\beta(l,m,n)=e^{-i(t_1\log l+t_2\log m+t_3\log n)}.$$

We have, by (4.6)

$$T_3(s_1,s_2,s_3;M,2M)$$
$$=\sum_{M<n_1\leqslant n_2\leqslant n_3\leqslant 2M}\frac{1}{n_1^{s_1}n_2^{s_2}n_3^{s_3}}-\sum_{M<n_1<n_3\leqslant 2M}\frac{1}{n_1^{s_1+s_2}n_3^{s_3}}$$
$$-\sum_{M<n_1<n_2\leqslant 2M}\frac{1}{n_1^{s_1}n_2^{s_2+s_3}}-\sum_{M<n_1\leqslant 2M}\frac{1}{n_1^{s_1+s_2+s_3}}$$
$$\ll\tau^2\log^2\tau\, M^{-\sigma_1-\sigma_2-\sigma_3},$$

and hence

$$(4.11)\qquad \sum_{j\leqslant j_0}T_3\bigl(s_1,s_2,s_3;2^{-j}\tau,2^{-j+1}\tau\bigr)\ll\tau^2L^{-\sigma_1-\sigma_2-\sigma_3}\log^3\tau$$
$$\ll\tau^{2-\frac{2}{3}(\sigma_1+\sigma_2+\sigma_3)}\log^3\tau.$$

On the other hand, for $M\leqslant N<2M\ll\tau^{2/3}$, we apply the theory of finite double zeta sums for the evaluation of $T_3(s_1, s_2, s_3; M, N)$. Since

$$T_3(s_1, s_2, s_3; M, N) = \sum_{M<n_3\leqslant N} \frac{1}{n_3^{s_3}} \left(\sum_{M<n_1<n_2<n_3} \frac{1}{n_1^{s_1} n_2^{s_2}} \right)$$
$$= \sum_{M<n_3\leqslant N} \frac{1}{n_3^{s_3}} T_2(s_1, s_2; M, n_3),$$

we have, by (2.8),

$$\text{(4.12)} \qquad T_3(s_1, s_2, s_3; M, N) \ll \begin{cases} M^{3-\sigma_1-\sigma_2-\sigma_3} \log \tau & (M \ll \tau^{\frac{1}{3}}) \\ M^{2-\sigma_1-\sigma_2-\sigma_3} \tau^{\frac{1}{3}} \log^{\frac{1}{2}} \tau & (\tau^{\frac{1}{3}} \ll M \ll \tau^{\frac{2}{3}}), \end{cases}$$

therefore, we have

$$\sum_{J>J_0} T_3\left(s_1, s_2, s_3; 2^{-J}\tau, 2^{-J+1}\tau\right) \ll \max\left\{\tau^{\frac{1}{3}-\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)}, \tau^{\frac{5}{3}-\frac{2}{3}(\sigma_1+\sigma_2+\sigma_3)}\right\} \log^2 \tau.$$

Combining (4.11) and the above, we have

$$\text{(4.13)} \qquad \sum_{J} T_3\left(s_1, s_2, s_3; 2^{-J}\tau, 2^{-J+1}\tau\right) \ll \tau^{2-\frac{2}{3}(\sigma_1+\sigma_2+\sigma_3)} \log^3 \tau.$$

Next we will treat the sums of $U_1(s_1, s_2, s_3; M)$ and $U_2(s_1, s_2, s_3; M)$. By (2.3), (2.8), (2.10), (4.8) and (4.9) we have

$$\text{(4.14)} \qquad \sum_{J\geqslant 1} U_1\left(s_1, s_2, s_3; 2^{-J}\tau\right)$$
$$\ll \begin{cases} \tau^{\frac{3}{2}-\frac{2}{3}(\sigma_1+\sigma_2+\sigma_3)} \log^{\frac{5}{2}} \tau & (\sigma_2+\sigma_3 \leqslant 1, 0 < \sigma_1 \leqslant \frac{1}{2}) \\ \tau^{\frac{7}{6}-\frac{1}{3}(2\sigma_1+\sigma_2+\sigma_3)} \log^{\frac{5}{2}} \tau & (\sigma_2+\sigma_3 > 1, 0 < \sigma_1 \leqslant \frac{1}{2}) \\ \tau^{\frac{4}{3}-\frac{1}{3}(\sigma_1+2\sigma_2+2\sigma_3)} \log^{\frac{7}{2}} \tau & (\sigma_2+\sigma_3 \leqslant 1, \frac{1}{2} < \sigma_1 \leqslant 1) \\ \tau^{1-\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)} \log^{\frac{7}{2}} \tau & (\sigma_2+\sigma_3 > 1, \frac{1}{2} < \sigma_1 \leqslant 1) \end{cases}$$

and

$$\text{(4.15)} \qquad \sum_{J\geqslant 1} U_2\left(s_1, s_2, s_3; 2^{-J}\tau\right)$$
$$\ll \begin{cases} \tau^{\frac{3}{2}-\frac{2}{3}(\sigma_1+\sigma_2+\sigma_3)} \log^3 \tau & (0 \leqslant \sigma_1 \leqslant \frac{1}{2}, 0 \leqslant \sigma_2 \leqslant \frac{1}{2}, 0 < \sigma_3 \leqslant \frac{1}{2}) \\ \tau^{\frac{4}{3}-\frac{1}{3}(2\sigma_1+2\sigma_2+\sigma_3)} \log^4 \tau & (0 \leqslant \sigma_1 \leqslant \frac{1}{2}, 0 \leqslant \sigma_2 \leqslant \frac{1}{2}, \frac{1}{2} < \sigma_3 \leqslant 1) \\ \tau^{\frac{4}{3}-\frac{1}{3}(\sigma_1+2\sigma_2+2\sigma_3)} \log^4 \tau & (\frac{1}{2} < \sigma_1 < 1, 0 \leqslant \sigma_2 \leqslant \frac{1}{2}, 0 \leqslant \sigma_3 \leqslant \frac{1}{2}) \\ \tau^{\frac{7}{6}-\frac{1}{3}(\sigma_1+2\sigma_2+\sigma_3)} \log^5 \tau & (\frac{1}{2} < \sigma_1 < 1, 0 \leqslant \sigma_2 \leqslant \frac{1}{2}, \frac{1}{2} < \sigma_3 \leqslant 1) \\ \tau^{\frac{4}{3}-\frac{1}{3}(2\sigma_1+\sigma_2+2\sigma_3)} \log^4 \tau & (0 \leqslant \sigma_1 \leqslant \frac{1}{2}, \frac{1}{2} < \sigma_2 < 1, 0 < \sigma_3 \leqslant \frac{1}{2}) \\ \tau^{\frac{7}{6}-\frac{1}{3}(2\sigma_1+\sigma_2+\sigma_3)} \log^5 \tau & (0 \leqslant \sigma_1 \leqslant \frac{1}{2}, \frac{1}{2} < \sigma_2 < 1, \frac{1}{2} < \sigma_3 \leqslant 1) \\ \tau^{\frac{7}{6}-\frac{1}{3}(\sigma_1+\sigma_2+2\sigma_3)} \log^5 \tau & (\frac{1}{2} < \sigma_1 < 1, \frac{1}{2} < \sigma_2 < 1, 0 \leqslant \sigma_3 \leqslant \frac{1}{2}) \\ \tau^{1-\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)} \log^6 \tau & (\frac{1}{2} < \sigma_1 < 1, \frac{1}{2} < \sigma_2 < 1, \frac{1}{2} < \sigma_3 \leqslant 1). \end{cases}$$

Hence, from (4.13), (4.14) and (4.15), we obtain

$$S_1^{(3)}(s_1, s_2, s_3; \tau) \ll \tau^{2-\frac{2}{3}(\sigma_1+\sigma_2+\sigma_3)} \log^3 \tau. \tag{4.16}$$

4.3. Evaluation of $S_2^{(3)}(s_1, s_2, s_3; \tau)$

In this subsection we prove

$$S_2^{(3)}(s_1, s_2, s_3; \tau) \ll \tau^{2-\sigma_1-\sigma_2-\sigma_3}. \tag{4.17}$$

We assume firstly that $\sigma_j > 1$ for $j = 1, 2, 3$, then the infinite sum $S_2^{(3)}(s_1, s_2, s_3; \tau)$ is absolutely convergent. We transform the sum $S_2^{(3)}(s_1, s_2, s_3; \tau)$ as follows:

$$\begin{aligned} S_2^{(3)}(s_1, s_2, s_3; \tau) &= \sum_{n_1<n_2\leqslant\tau} \frac{1}{n_1^{s_1} n_2^{s_2}} \sum_{n_3>\tau} \frac{1}{n_3^{s_3}} + \sum_{n_1\leqslant\tau} \frac{1}{n_1^{s_1}} \sum_{\tau<n_2<n_3} \frac{1}{n_2^{s_2} n_3^{s_3}} \\ &\quad + \sum_{\tau<n_1<n_2<n_3} \frac{1}{n_1^{s_1} n_2^{s_2} n_3^{s_3}} \\ &= S_{2,1}^{(3)}(s_1, s_2, s_3; \tau) \\ &\quad + S_{2,2}^{(3)}(s_1, s_2, s_3; \tau) + S_{2,3}^{(3)}(s_1, s_2, s_3; \tau), \end{aligned} \tag{4.18}$$

say.

We consider $S_{2,3}^{(3)}(s_1, s_2, s_3; \tau)$. Applying Lemma 1 for the sum on n_3, we get

$$\begin{aligned} \sum_{\tau<n_2<n_3} \frac{1}{n_2^{s_2} n_3^{s_3}} &= \frac{1}{s_3-1} \sum_{\tau<n_2} \frac{1}{n_2^{s_2+s_3-1}} - \frac{1}{2} \sum_{\tau<n_2} \frac{1}{n_2^{s_2+s_3}} \\ &\quad + \frac{s_3}{12} \sum_{\tau<n_2} \frac{1}{n_2^{s_2+s_3+1}} + O\left(\tau^3 \sum_{\tau<n_2} \frac{1}{n_2^{\sigma_2+\sigma_3+3}}\right) \\ &=: \sum_{j=1}^{4} J_j(s_2, s_3; \tau). \end{aligned} \tag{4.19}$$

For J_1, using Lemma 1 for the sum on n_2 again, we get

$$\begin{aligned} J_1(s_2, s_3; \tau) &= \frac{\tau^{2-s_2-s_3}}{(s_3-1)(s_2+s_3-2)} - \frac{\tau^{1-s_2-s_3}}{2(s_3-1)} + \frac{(s_2+s_3-1)\tau^{-s_2-s_3}}{12(s_3-1)} \\ &\quad + O\left(\frac{|t_2+t_3|^3 \tau^{-\sigma_2-\sigma_3-2}}{|s_3-1|}\right). \end{aligned}$$

This expression holds true for $\sigma_2 + \sigma_3 > -2$, hence by analytic continuation, we have

$$J_1(s_2, s_3; \tau) \ll \tau^{1-\sigma_2-\sigma_3}$$

for $0 \leqslant \sigma_2, \sigma_3 < 1$ and $|s_2 + s_3 - 2| \gg 1$. Similarly we have

$$J_j(s_2, s_3; \tau) \ll \tau^{1-\sigma_2-\sigma_3} \quad (j = 2, 3, 4)$$

for $0 \leqslant \sigma_2, \sigma_3 < 1$, $|s_2 + s_3 - 1| \gg 1$ and $|s_2 + s_3| \gg 1$. From (2.2), (2.3) and the above estimates, we have

$$S_{2,3}^{(3)}(s_1, s_2, s_3; \tau) \ll \tau^{2-\sigma_1-\sigma_2-\sigma_3} \tag{4.20}$$

for $0 \leqslant \sigma_j < 1$ $(j = 1, 2, 3)$ and $|t_2 + t_3| \gg 1$.

For $S_{2,3}^{(3)}(s_1, s_2, s_3; \tau)$, we use Lemma 1 for the sum on n_3, n_2 and n_1 successively and can deduce that

$$S_{2,3}^{(3)}(s_1, s_2, s_3; \tau) \ll \tau^{2-\sigma_1-\sigma_2-\sigma_3}$$

for $|t_2 + t_3| \gg 1$ and $|t_1 + t_2 + t_3| \gg 1$.

Finally we treat $S_{2,1}^{(3)}(s_1, s_2, s_3; \tau)$. Since $S_{2,1}^{(3)}(s_1, s_2, s_3; \tau) = S_1^{(2)}(s_1, s_2; \tau) \times \sum_{\tau < n_3} \frac{1}{n_3^{s_3}}$, where $S_1^{(2)}(s_1, s_2; \tau)$ is defined in (2.5), we can use the estimate (2.10). Considering all cases there, it turned out that

$$S_{2,1}^{(3)}(s_1, s_2, s_3; \tau) \ll \tau^{2-\sigma_1-\sigma_2-\sigma_3}. \tag{4.21}$$

Though we can get the better estimate for $S_{2,1}^{(3)}$, (4.21) is enough if we count the estimate for $S_{2,2}^{(3)}$ and $S_{2,3}^{(3)}$. We also note that for the estimate (4.21) we do not need the assumption $|t_1 + t_2| \gg 1$. Hence we obtain (4.17).

4.4. Proof of Theorem

The result (1.7) now follows from (4.1) and taking $\sigma_1 = \sigma_2 = \sigma_3 = 0$ into (4.16) and (4.17). From (4.1), (4.16) and (4.17), we get the assertion (1.8).

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